MATH 2028 Honours Advanced Calculus II
2021-22 Term 1
Problem Set 10
due on Nov 29, 2021 (Monday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Gradescope on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Notations:We will use $\mathcal{A}^k(\mathbb{R}^n)$ to denote space of differential k-forms on $\mathbb{R}^n$.

Problems to hand in

1. Let $n = (n_1, n_2, n_3) \in \mathbb{R}^3$ be a unit vector and $v, w \in \mathbb{R}^3$ be orthogonal to n. Let
$$\omega = n_1\, dy \wedge dz + n_2\, dz \wedge dx + n_3\, dx \wedge dy.$$
Prove that $\omega(v, w)$ is the signed area of the parallelogram spanned by v and w (the sign being determined by whether $\{n, v, w\}$ forms a right-handed orthonormal basis for $\mathbb{R}^3$).

2. Let $g(\rho, \phi, \theta) : (0, \infty) \times (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$ be the spherical coordinates map, i.e.
$$g(\rho, \phi, \theta) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi).$$
Compute $g^*(dx \wedge dy \wedge dz)$.

3. We say that a k-form is *closed* if $d\omega = 0$ and *exact* if $\omega = d\eta$ for some $(k-1)$-form η.
   (a) Prove that an exact form is closed. Is every closed form exact?
   (b) Prove that if ω and ϕ are closed, then $\omega \wedge \phi$ is closed.
   (c) Prove that if ω is exact and ϕ is closed, then $\omega \wedge \phi$ is exact.

Suggested Exercises

1. Suppose $\omega \in \Lambda^k(\mathbb{R}^n)^*$ and k is odd. Prove that $\omega \wedge \omega = 0$. Give an example to show that it does no hold when k is even.

2. Let $v, w \in \mathbb{R}^3$. Prove that $dx(v \times w) = dy \wedge dz(v, w)$, $dy(v \times w) = dz \wedge dx(v, w)$ and $dz(v \times w) = dx \wedge dy(v, w)$.

3. Can there be a function f so that df is the given 1-form ω (everywhere ω is defined)? If so, find f.
   (a) $\omega = -y\, dx + x\, dy$
   (b) $\omega = 2xy\, dx + x^2\, dy$
   (c) $\omega = y\, dx + z\, dy + x\, dz$
   (d) $\omega = (x^2 + yz)\, dx + (xz + \cos y)\, dy + (z + xy)\, dz$

(e) $\omega = \frac{x}{x^2+y^2}\, dx + \frac{y}{x^2+y^2}\, dy$

(f) $\omega = -\frac{y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$

4. For each of the following k-forms ω, can there be a $(k-1)$-form η (defined wherever ω is) so that $d\eta = \omega$?

   (a) $\omega = dx \wedge dy$

   (b) $\omega = x\, dx \wedge dy$

   (c) $\omega = z\, dx \wedge dy$

   (d) $\omega = z\, dx \wedge dy + y\, dx \wedge dz + z\, dy \wedge dz$

   (e) $\omega = x\, dy \wedge dz + y\, dx \wedge dz + z\, dx \wedge dy$

   (f) $\omega = (x^2 + y^2 + z^2)^{-1}(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy)$

5. Define $* : \mathcal{A}^1(\mathbb{R}^3) \to \mathcal{A}^2(\mathbb{R}^3)$ by

$$*(dx) = dy \wedge dz, \quad *(dy) = dz \wedge dx \quad \text{and} \quad *(dz) = dx \wedge dy,$$

   extending by linearity. If f is a smooth function, show that

$$d * (df) = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) dx \wedge dy \wedge dz.$$

6. Suppose $\omega \in \mathcal{A}^1(\mathbb{R}^n)$ and theres is a nowhere-zero function λ so that $\lambda\omega = df$ for some function f. Prove that $\omega \wedge d\omega = 0$.

7. In each of the following, compute the pullback $g^*\omega$ and verify that $g^*(d\omega) = d(g^*\omega)$:

   (a) $g(v) = (3\cos 2v, 3\sin 2v)$, $\omega = -y\, dx + x\, dy$

   (b) $g(u, v) = (\cos u, \sin u, v)$, $\omega = z\, dx + x\, dy + y\, dz$

   (c) $g(u, v) = (\cos u, \sin v, \sin u, \cos v)$, $\omega = (-x_3\, dx_1 + x_1\, dx_3) \wedge (-x_2\, dx_4 + x_4\, dx_2)$

8. Suppose that $k \leq n$. Let $\omega_1, \cdots, \omega_k \in (\mathbb{R}^n)^*$ and suppose that $\sum_{i=1}^k dx_i \wedge \omega_i = 0$. Prove that there exist $a_{ij} \in \mathbb{R}$ such that $a_{ji} = a_{ij}$ and $\omega_i = \sum_{j=1}^k a_{ij} dx_j$.

9. Suppose $U \subset \mathbb{R}^m$ is open and $g : U \to \mathbb{R}^n$ is smooth. Prove that for any $\omega \in \mathcal{A}^k(\mathbb{R}^n)$ and $v_1, \cdots, v_k \in \mathbb{R}^m$, we have

$$g^*\omega(a)(v_1, \cdots, v_k) = \omega(g(a))(Dg(a)v_1, \cdots, Dg(a)v_k).$$

Challenging Exercises

1. Prove that there is a unique linear operator $d : \mathcal{A}^k(\mathbb{R}^n) \to \mathcal{A}^{k+1}(\mathbb{R}^n)$ for all k such that

   (1) $df = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\, dx_j$ for all functions $f : \mathbb{R}^n \to \mathbb{R}$

   (2) $d(f\omega) = df \wedge \omega + f\, d\omega$ for all functions $f : \mathbb{R}^n \to \mathbb{R}$ and $\omega \in \mathcal{A}^k(\mathbb{R}^n)$

   (3) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for any $\omega \mathcal{A}^k(\mathbb{R}^n)$, $\eta \in \mathcal{A}^\ell(\mathbb{R}^n)$

   (4) $d(d\omega) = 0$ for all $\omega \in \mathcal{A}^k(\mathbb{R}^n)$